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5. Harmonic analysis on algebraic groups over two-dimensional local fields of equal characteristic

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In this section we review the main parts of a recent work [4] on harmonic analysis on algebraic groups over two-dimensional local fields.

5.1. Groups and buildings

Let K ($K = K_2$ whose residue field is K_1 whose residue field is K_0, see the notation in section 1 of Part I) be a two-dimensional local field of equal characteristic. Thus K_2 is isomorphic to the Laurent series field $K_1((t_2))$ over K_1. It is convenient to think of elements of K_2 as (formal) loops over K_1. Even in the case where $\mathrm{char}(K_1) = 0$, it is still convenient to think of elements of K_1 as (generalized) loops over K_0 so that K_2 consists of double loops.

Denote the residue map $\mathcal{O}_{K_2} \to K_1$ by p_2 and the residue map $\mathcal{O}_{K_1} \to K_0$ by p_1. Then the ring of integers O_K of K as of a two-dimensional local field (see subsection 1.1 of Part I) coincides with $p_2^{-1}(\mathcal{O}_{K_1})$.

Let G be a split simple simply connected algebraic group over $\mathbb{Z}$ (e.g. $G = SL_2$). Let $T \subset B \subset G$ be a fixed maximal torus and Borel subgroup of G; put $N = [B, B]$, and let W be the Weyl group of G. All of them are viewed as group schemes.

Let $L = \mathrm{Hom}(\mathbb{G}_m, T)$ be the coweight lattice of G; the Weyl group acts on L.

Recall that $I(K_1) = p_1^{-1}(B(\mathbb{F}_q))$ is called an Iwahori subgroup of $G(K_1)$ and $T(\mathcal{O}_{K_1})N(K_1)$ can be seen as the "connected component of unity" in $B(K_1)$. The latter name is explained naturally if we think of elements of $B(K_1)$ as being loops with values in B.

Definition. Put

$$D_0 = p_2^{-1} p_1^{-1}(B(\mathbb{F}_q)) \subset G(O_K),$$
$$D_1 = p_2^{-1}(T(\mathcal{O}_{K_1})N(K_1)) \subset G(O_K),$$
$$D_2 = T(\mathcal{O}_{K_2})N(K_2) \subset G(K).$$

Then D_2 can be seen as the "connected component of unity" of $B(K)$ when K is viewed as a two-dimensional local field, D_1 is a (similarly understood) connected component of an Iwahori subgroup of $G(K_2)$, and D_0 is called a double Iwahori subgroup of $G(K)$.

A choice of a system of local parameters t_1, t_2 of K determines the identification of the group K^*/O_K^* with $\mathbb{Z} \oplus \mathbb{Z}$ and identification $L \oplus L$ with $L \otimes (K^*/O_K^*)$.

We have an embedding of $L \otimes (K^*/O_K^*)$ into $T(K)$ which takes $a \otimes (t_1^j t_2^j)$, $i, j \in \mathbb{Z}$, to the value on $t_1^i t_2^j$ of the 1-parameter subgroup in T corresponding to a.

Define the action of W on $L \otimes (K^*/O_K^*)$ as the product of the standard action on L and the trivial action on K^*/O_K^*. The semidirect product

$$\widehat{\widehat{W}} = (L \otimes K^*/O_K^*) \rtimes W$$

is called the *double affine Weyl group* of G.

A (set-theoretical) lifting of W into $G(O_K)$ determines a lifting of $\widehat{\widehat{W}}$ into $G(K)$.

Proposition. *For every $i, j \in \{0, 1, 2\}$ there is a disjoint decomposition*

$$G(K) = \dot{\bigcup}_{w \in \widehat{\widehat{W}}} D_i w D_j .$$

The identification $D_i \backslash G(K)/D_j$ with $\widehat{\widehat{W}}$ doesn't depend on the choice of liftings.

Proof. Iterated application of the Bruhat, Bruhat–Tits and Iwasawa decompositions to the local fields K_2, K_1.

For the Iwahori subgroup $I(K_2) = p_2^{-1}(B(K_1))$ of $G(K_2)$ the homogeneous space $G(K)/I(K_2)$ is the "affine flag variety" of G, see [5]. It has a canonical structure of an ind-scheme, in fact, it is an inductive limit of projective algebraic varieties over K_1 (the closures of the affine Schubert cells).

Let $B(G, K_2/K_1)$ be the *Bruhat–Tits building* associated to G and the field K_2. Then the space $G(K)/I(K_2)$ is a $G(K)$-orbit on the set of flags of type (vertex, maximal cell) in the building. For every vertex v of $B(G, K_2/K_1)$ its locally finite Bruhat–Tits building β_v isomorphic to $B(G, K_1/K_0)$ can be viewed as a "microbuilding" of the *double Bruhat–Tits building* $B(G, K_2/K_1/K_0)$ of K as a two-dimensional local field constructed by Parshin ([7], see also section 3 of Part II). Then the set $G(K)/D_1$ is identified naturally with the set of all the horocycles $\{w \in \beta_v : d(z, w) = r\}$, $z \in \partial\beta_v$ of the microbuildings β_v (where the "distance" $d(z, \ )$ is viewed as an element of

a natural L-torsor). The fibres of the projection $G(K)/D_1 \to G(K)/I(K_2)$ are L-torsors.

5.2. The central extension and the affine Heisenberg–Weyl group

According to the work of Steinberg, Moore and Matsumoto [6] developed by Brylinski and Deligne [1] there is a central extension

$$1 \to K_1^* \to \Gamma \to G(K_2) \to 1$$

associated to the tame symbol $K_2^* \times K_2^* \to K_1^*$ for the couple (K_2, K_1) (see subsection 6.4.2 of Part I for the general definition of the tame symbol).

Proposition. *This extension splits over every* D_i, $0 \leqslant i \leqslant 2$.

Proof. Use Matsumoto's explicit construction of the central extension.

Thus, there are identifications of every D_i with a subgroup of Γ. Put

$$\Delta_i = \mathcal{O}_{K_1}^* D_i \subset \Gamma, \qquad \Xi = \Gamma/\Delta_1.$$

The minimal integer scalar product Ψ on L and the composite of the tame symbol $K_2^* \times K_2^* \to K_1^*$ and the discrete valuation $v_{K_1}\colon K^* \to \mathbb{Z}$ induces a W-invariant skew-symmetric pairing $L \otimes K^*/O_K^* \times L \otimes K^*/O_K^* \to \mathbb{Z}$. Let

$$1 \to \mathbb{Z} \to \mathcal{L} \to L \otimes K^*/O_K^* \to 1$$

be the central extension whose commutator pairing corresponds to the latter skew-symmetric pairing. The group $\mathcal{L}$ is called the *Heisenberg group*.

Definition. The semidirect product

$$\widetilde{W} = \mathcal{L} \rtimes W$$

is called the *double affine Heisenberg–Weyl* group of G.

Theorem. *The group* $\widetilde{W}$ *is isomorphic to* $L_{\text{aff}} \rtimes \widehat{W}$ *where* $L_{\text{aff}} = \mathbb{Z} \oplus L$, $\widehat{W} = L \rtimes W$ *and*

$$w \circ (a, l') = (a, w(l)), \quad l \circ (a, l') = (a + \Psi(l, l'), l'), \qquad w \in W, \quad l, l' \in L, \quad a \in \mathbb{Z}.$$

For every $i, j \in \{0, 1, 2\}$ *there is a disjoint union*

$$\Gamma = \dot{\bigcup}_{w \in \widetilde{W}} \Delta_i w \Delta_j$$

and the identification $\Delta_i \backslash \Gamma / \Delta_j$ *with* $\widetilde{W}$ *is canonical.*

5.3. Hecke algebras in the classical setting

Recall that for a locally compact group Γ and its compact subgroup Δ the Hecke algebra $\mathcal{H}(\Gamma,\Delta)$ can be defined as the algebra of compactly supported double Δ-invariant continuous functions of Γ with the operation given by the convolution with respect to the Haar measure on Γ. For $C = \Delta\gamma\Delta \in \Delta\backslash\Gamma/\Delta$ the Hecke correspondence $\Sigma_C = \{(\alpha\Delta,\beta\Delta) : \alpha\beta^{-1} \in C\}$ is a Γ-orbit of $(\Gamma/\Delta)\times(\Gamma/\Delta)$.

For $x \in \Gamma/\Delta$ put $\Sigma_C(x) = \Sigma_C \cap (\Gamma/\Delta)\times\{x\}$. Denote the projections of Σ_C to the first and second component by π_1 and π_2.

Let $\mathcal{F}(\Gamma/\Delta)$ be the space of continuous functions $\Gamma/\Delta \to \mathbb{C}$. The operator

$$\tau_C\colon \mathcal{F}(\Gamma/\Delta) \to \mathcal{F}(\Gamma/\Delta), \quad f \to \pi_{2*}\pi_1^*(f)$$

is called the *Hecke operator* associated to C. Explicitly,

$$(\tau_C f)(x) = \int_{y\in\Sigma_C(x)} f(y)d\mu_{C,x},$$

where $\mu_{C,x}$ is the $\mathrm{Stab}(x)$-invariant measure induced by the Haar measure. Elements of the Hecke algebra $\mathcal{H}(\Gamma,\Delta)$ can be viewed as "continuous" linear combinations of the operators τ_C, i.e., integrals of the form $\int \phi(C)\tau_C dC$ where dC is some measure on $\Delta\backslash\Gamma/\Delta$ and ϕ is a continuous function with compact support. If the group Δ is also open (as is usually the case in the p-adic situation), then $\Delta\backslash\Gamma/\Delta$ is discrete and $\mathcal{H}(\Gamma,\Delta)$ consists of finite linear combinations of the τ_C.

5.4. The regularized Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$

Since the two-dimensional local field K and the ring O_K are not locally compact, the approach of the previous subsection would work only after a new appropriate integration theory is available.

The aim of this subsection is to make sense of the Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$.

Note that the fibres of the projection $\Xi = \Gamma/\Delta_1 \to G(K)/I(K_2)$ are L_{aff}-torsors and $G(K)/I(K_2)$ is the inductive limit of compact (profinite) spaces, so Ξ can be considered as an object of the category $\mathcal{F}_1$ defined in subsection 1.2 of the paper of Kato in this volume.

Using Theorem of 5.2 for $i = j = 1$ we introduce:

Definition. For $(w,l) \in \widetilde{W} = L_{\mathrm{aff}} \rtimes \widehat{W}$ denote by $\Sigma_{w,l}$ the Hecke correspondence (i.e., the Γ-orbit of $\Xi\times\Xi$) associated to (w,l). For $\xi \in \Xi$ put

$$\Sigma_{w,l}(\xi) = \{\xi' : (\xi,\xi') \in \Sigma_{w,l}\}.$$

The stabilizer $\mathrm{Stab}(\xi) \leqslant \Gamma$ acts transitively on $\Sigma_{w,l}(\xi)$.

Proposition. *$\Sigma_{w,l}(\xi)$ is an affine space over K_1 of dimension equal to the length of $w \in \widehat{W}$. The space of compex valued Borel measures on $\Sigma_{w,l}(\xi)$ is 1-dimensional. A choice of a $\mathrm{Stab}(\xi)$-invariant measure $\mu_{w,l,\xi}$ on $\Sigma_{w,l}(\xi)$ determines a measure $\mu_{w,l,\xi'}$ on $\Sigma_{w,l}(\xi')$ for every ξ'.*

Definition. For a continuous function $f\colon \Xi \to \mathbb{C}$ put

$$(\tau_{w,l} f)(\xi) = \int_{\eta\in\Sigma_{w,l}(\xi)} f(\eta) d\mu_{w,l,\xi}.$$

Since the domain of the integration is not compact, the integral may diverge. As a first step, we define the space of functions on which the integral makes sense. Note that Ξ can be regarded as an L_{aff}-torsor over the ind-object $G(K)/I(K_2)$ in the category $\mathrm{pro}(C_0)$, i.e., a compatible system of L_{aff}-torsors Ξ_ν over the affine Schubert varieties Z_ν forming an exhaustion of $G(K)/I(K_1)$. Each Ξ_ν is a locally compact space and Z_ν is a compact space. In particular, we can form the space $\mathcal{F}_0(\Xi_\nu)$ of locally constant complex valued functions on Ξ_ν whose support is compact (or, what is the same, proper with respect to the projection to Z_ν). Let $\mathcal{F}(\Xi_\nu)$ be the space of all locally constant complex functions on Ξ_ν. Then we define $\mathcal{F}_0(\Xi) = \text{"}\varprojlim\text{"}\, \mathcal{F}_0(\Xi_\nu)$ and $\mathcal{F}(\Xi) = \text{"}\varprojlim\text{"}\, \mathcal{F}(\Xi_\nu)$. They are pro-objects in the category of vector spaces. In fact, because of the action of L_{aff} and its group algebra $\mathbb{C}[L_{\text{aff}}]$ on Ξ, the spaces $\mathcal{F}_0(\Xi), \mathcal{F}(\Xi)$ are naturally pro-objects in the category of $\mathbb{C}[L_{\text{aff}}]$-modules.

Proposition. *If $f = (f_\nu) \in \mathcal{F}_0(X)$ then $\mathrm{Supp}(f_\nu) \cap \Sigma_{w,l}(\xi)$ is compact for every w, l, ξ, ν and the integral above converges. Thus, there is a well defined Hecke operator*

$$\tau_{w,l}\colon \mathcal{F}_0(\Xi) \to \mathcal{F}(\Xi)$$

which is an element of $\mathrm{Mor}(\mathrm{pro}(\mathrm{Mod}_{\mathbb{C}[L_{\text{aff}}]}))$. In particular, $\tau_{w,l}$ is the shift by l and $\tau_{w,l+l'} = \tau_{w,l'}\tau_{e,l}$.

Thus we get Hecke operators as operators from one (pro-)vector space to another, bigger one. This does not yet allow to compose the $\tau_{w,l}$. Our next step is to consider certain infinite linear combinations of the $\tau_{w,l}$.

Let $T^\vee_{\text{aff}} = \mathrm{Spec}(\mathbb{C}[L_{\text{aff}}])$ be the "dual affine torus" of G. A function with finite support on L_{aff} can be viewed as the collection of coefficients of a polynomial, i.e., of an element of $\mathbb{C}[L_{\text{aff}}]$ as a regular function on $T^\vee_{\text{aff}}$. Further, let $Q \subset L_{\text{aff}} \otimes \mathbb{R}$ be a strictly convex cone with apex 0. A function on L_{aff} with support in Q can be viewed as the collection of coefficients of a formal power series, and such series form a ring containing $\mathbb{C}[L_{\text{aff}}]$. On the level of functions the ring operation is the convolution. Let $\mathcal{F}_Q(L_{\text{aff}})$ be the space of functions whose support is contained in some translation of Q. It is a ring with respect to convolution.

Let $\mathbb{C}(L_{\text{aff}})$ be the field of rational functions on $T^\vee_{\text{aff}}$. Denote by $F^{\text{rat}}_Q(L_{\text{aff}})$ the subspace in $F_Q(L_{\text{aff}})$ consisting of functions whose corresponding formal power series are expansions of rational functions on $T^\vee_{\text{aff}}$.

If A is any L_{aff}-torsor (over a point), then $\mathcal{F}_0(A)$ is an (invertible) module over $\mathcal{F}_0(L_{\text{aff}}) = \mathbb{C}[L_{\text{aff}}]$ and we can define the spaces $\mathcal{F}_Q(A)$ and $\mathcal{F}^{\text{rat}}_Q(A)$ which will be modules over the corresponding rings for L_{aff}. We also write $\mathcal{F}^{\text{rat}}(A) = \mathcal{F}_0(A) \otimes_{\mathbb{C}[L_{\text{aff}}]} \mathbb{C}(L_{\text{aff}})$.

We then extend the above concepts "fiberwise" to torsors over compact spaces (objects of $\text{pro}(C_0)$) and to torsors over objects of $\text{ind}(\text{pro}(C_0))$ such as Ξ.

Let $w \in \widehat{W}$. We denote by $Q(w)$ the image under w of the cone of dominant affine coweights in L_{aff}.

Theorem. *The action of the Hecke operator $\tau_{w,l}$ takes $\mathcal{F}_0(\Xi)$ into $\mathcal{F}^{\text{rat}}_{Q(w)}(\Xi)$. These operators extend to operators*

$$\tau^{\text{rat}}_{w,l} : \mathcal{F}^{\text{rat}}(\Xi) \to \mathcal{F}^{\text{rat}}(\Xi).$$

Note that the action of $\tau^{\text{rat}}_{w,l}$ involves a kind of regularization procedure, which is hidden in the identification of the $\mathcal{F}^{\text{rat}}_{Q(w)}(\Xi)$ for different w, with subspaces of the same space $\mathcal{F}^{\text{rat}}(\Xi)$. In practical terms, this involves summation of a series to a rational function and re-expansion in a different domain.

Let $\mathcal{H}_{\text{pre}}$ be the space of finite linear combinations $\sum_{w,l} a_{w,l}\tau_{w,l}$. This is not yet an algebra, but only a $\mathbb{C}[L_{\text{aff}}]$-module. Note that elements of $\mathcal{H}_{\text{pre}}$ can be written as finite linear combinations $\sum_{w\in\widehat{W}} f_w(t)\tau_w$ where $f_w(t) = \sum_l a_{w,l}t^l$, $t \in T^\vee_{\text{aff}}$, is the polynomial in $\mathbb{C}[L_{\text{aff}}]$ corresponding to the collection of the $a_{w,l}$. This makes the $\mathbb{C}[L_{\text{aff}}]$-module structure clear. Consider the tensor product

$$\mathcal{H}_{\text{rat}} = \mathcal{H}_{\text{pre}} \otimes_{\mathbb{C}[L_{\text{aff}}]} \mathbb{C}(L_{\text{aff}}).$$

Elements of this space can be considered as finite linear combinations $\sum_{w\in\widehat{W}} f_w(t)\tau_w$ where $f_w(t)$ are now rational functions. By expanding rational functions in power series, we can consider the above elements as certain infinite linear combinations of the $\tau_{w,l}$.

Theorem. *The space $\mathcal{H}_{\text{rat}}$ has a natural algebra structure and this algebra acts in the space $\mathcal{F}^{\text{rat}}(\Xi)$, extending the action of the $\tau_{w,l}$ defined above.*

The operators associated to $\mathcal{H}_{\text{rat}}$ can be viewed as certain integro-difference operators, because their action involves integration (as in the definition of the $\tau_{w,l}$) as well as inverses of linear combinations of shifts by elements of L (these combinations act as difference operators).

Definition. The regularized Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$ is, by definition, the subalgebra in $\mathcal{H}_{\text{rat}}$ consisting of elements whose action in $\mathcal{F}_{\text{rat}}(\Xi)$ preserves the subspace $\mathcal{F}_0(\Xi)$.

5.5. The Hecke algebra and the Cherednik algebra

In [2] I. Cherednik introduced the so-called double affine Hecke algebra Cher_q associated to the root system of G. As shown by V. Ginzburg, E. Vasserot and the author [3], Cher_q can be thought as consisting of finite linear combinations $\sum_{w\in\widehat{W}_{\mathrm{ad}}} f_w(t)[w]$ where W_{ad} is the affine Weyl group of the adjoint quotient G_{ad} of G (it contains $\widehat{W}$) and $f_w(t)$ are rational functions on $T^{\vee}_{\mathrm{aff}}$ satisfying certain residue conditions. We define the modified Cherednik algebra $\ddot{H}_q$ to be the subalgebra in Cher_q consisting of linear combinations as above, but going over $\widehat{W}\subset\widehat{W}_{\mathrm{ad}}$.

Theorem. *The regularized Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$ is isomorphic to the modified Cherednik algebra $\ddot{H}_q$. In particular, there is a natural action of $\ddot{H}_q$ on $\mathcal{F}_0(\Xi)$ by integro-difference operators.*

Proof. Use the principal series intertwiners and a version of Mellin transform. The information on the poles of the intertwiners matches exactly the residue conditions introduced in [3].

Remark. The only reason we needed to assume that the 2-dimensional local field K has equal characteristic was because we used the fact that the quotient $G(K)/I(K_2)$ has a structure of an inductive limit of projective algebraic varieties over K_1. In fact, we really use only a weaker structure: that of an inductive limit of profinite topological spaces (which are, in this case, the sets of K_1-points of affine Schubert varieties over K_1). This structure is available for any 2-dimensional local field, although there seems to be no reference for it in the literature. Once this foundational matter is established, all the constructions will go through for any 2-dimensional local field.

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